

On loop quantum gravity kinematics with non-degenerate spatial background

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Abstract

In a remarkable paper, T. Koslowski introduced kinematical representations for loop quantum gravity in which there is a non-degenerate spatial background metric present. He also considered their properties, and showed that Gauß and diffeomorphism constraints can be implemented. With the present article, we streamline and extend his treatment. In particular, we show that the standard regularization of the geometric operators leads to well defined operators in the new representations, and we work out their properties fully. We also give details on the implementation of the constraints. All of this is done in such a way as to show that the standard representation is a particular (and in some ways exceptional) case of the more general constructions. This does not mean that these new representations are as fundamental as the standard one. Rather, we believe they might be useful to find some form of effective theory of loop quantum gravity on large scales.

1 Introduction

In the paper [1], T. Koslowski introduced kinematical representations for loop quantum gravity in which there is a non-degenerate spatial background metric present. In these representations, the flux-operators of loop quantum gravity (encoding quantized spatial geometry) acquire a c-number term in addition to the standard one:

$$\pi(E_{S,f}) = X_{S,f} + E_{S,f}^{(0)} \text{id}, \quad \text{with } E_{S,f}^{(0)} = \int_S *E^{(0)} \lrcorner f_I. \quad (1.1)$$

The quantity $E_{S,f}^{(0)}$ is exactly the classical value of the flux in a background geometry given by a densitized triad $E^{(0)}$. The possibility of such representations was known to some experts before [1] (they were mentioned for example in passing in [2]) but they were not considered interesting, because they did not alleviate the asymmetry in fluctuations between the canonical variables, and because gauge transformations and diffeomorphisms could not be implemented unitarily. The remarkable discovery of [1] is that the transformations *can* be implemented if one is willing to go to a large direct sum of such representations, and that, even better, the corresponding constraints can then be implemented.¹

The goal of the present article is to round off the picture that is emerging in [1], by showing that geometric operators can be defined in the new representations *exactly* as in the standard representation, and that they have a very simple structure. For example we find for the volume operator of a region R that

$$V_R = V_R^{\text{vac}} + V_R^{(0)} \text{id} \quad (1.2)$$

¹It may be interesting to compare this to the situation in [3], where another new and interesting representation for loop quantum gravity is constructed, which is likewise highly reducible.

where $V_R^{(0)}$ is the classical volume of R in the spatial background geometry. Additionally we discuss the implementation of Gauß and diffeomorphism constraint, using exactly the same techniques as in the standard representation. This adds some detail and clarifications to the treatment in [1]. Finally [1] was using an algebra for loop quantum gravity that many researchers may not be very familiar with. Here we will work with the standard formalism.

As for the significance of these representations, we believe that they are not as fundamental as the standard representation. The latter shows how geometry can emerge from literally nothing, and it beautifully explains the black hole entropy-area relation, and more. The new representations, by contrast, are based on a classical geometry as input. Still they may be useful in some form of perturbative calculation, in which a huge number of elementary geometric excitations are subsumed into the classical background.

We should point out that the new representations are treating the two canonically conjugate quantities A and E very differently, in much the same way as the standard representation does. The states in the new representations, when viewed as functions of E , are sharply peaked around a classical spatial geometry, whereas they are almost constant when viewed as functions of the conjugate variable A . In this sense they fail to approximately describe a classical *space-time*. They only approximately describe a classical *spatial* geometry. While it would be very interesting to distribute the quantum mechanical uncertainty more evenly, and thus introduce also a classical background connection, the states considered here do not do that, and it is also not straightforward to generalize them in such a way that they do. It is however conceivable that they can be taken as the ingredients of some form of limiting procedure, to obtain states that encode a full space-time geometry.

We should also remark that the new representations, as well as those obtained from them for the purpose of unitarily implementing the spatial diffeomorphisms are *not* counterexamples to the uniqueness theorems [4, 5] on representations of the kinematical algebra. In both cases, crucial hypotheses are violated: As we will see in the next section, there is no unitary implementation of the action of spatial diffeomorphisms on the basic variables in the new representations. And in the larger spaces that do admit such a unitary implementation, the representation is highly reducible.

In the next section, we recall the definition of the new representations from [1] and state some elementary properties. Then we investigate the geometric operators in section 3. In sections 4 and 5 we discuss implementations of gauge transformations and diffeomorphisms, and of the constraints. We finish with some discussion and outlook.

2 The new kinematical representations

To fix notation, we will start by reviewing the standard LQG quantum kinematics. The canonical pair to be quantized consists of an $SU(2)$ connection A and a densitized triad field E . These fields transform under the automorphisms of the $SU(2)$ bundle. Upon choosing (local) trivializations, these bundle automorphisms split into gauge transformations and diffeomorphisms. The automorphism group is then the semi-direct product of those subgroups,

$$(g, \phi) \cdot (g', \phi') = (g\alpha_\phi(g'), \phi \circ \phi'), \quad (2.1)$$

with g, g' gauge transformations, ϕ, ϕ' diffeomorphisms, and the automorphism $\alpha_\phi(g) = \phi_*g$ on the group of gauge transformations given by the push-forward under diffeomorphisms. The fields then transform under gauge transformations g and diffeomorphisms ϕ in the standard way:

$$A \xrightarrow{g} \text{ad}_g(A) + g^{-1}dg, \quad A \xrightarrow{\phi} \phi_*A \quad (2.2)$$

$$E \xrightarrow{g} \text{ad}_g(E) =: g \triangleright E, \quad E \xrightarrow{\phi} \phi_*E =: \phi \triangleright E \quad (2.3)$$

where ad is the adjoint action of $\text{SU}(2)$ on $\mathfrak{su}(2)$, and the star signifies push-forward. The basic variables used for quantization are chosen in such a way as to make their transformation behavior under $\text{SU}(2)$ and spatial diffeomorphisms as simple and transparent as possible. For the connection, one considers holonomies

$$h_\alpha[A] = \mathcal{P} \exp \int_\alpha A, \quad (2.4)$$

or more generally, functions of such holonomies,

$$T[A] \equiv t(h_{\alpha_1}[A], h_{\alpha_2}[A], \dots, h_{\alpha_n}[A]) \quad (2.5)$$

for a finite number of paths $\alpha_1, \dots, \alpha_n$ forming a graph, and a function t on couples of $\text{SU}(2)$. Such functionals are also called *cylindrical functions*.

For the field E a natural functional is its flux through surfaces S :

$$E_{S,f}[E] = \int_S *E_I f^I \quad (2.6)$$

where f is a function taking values in $\mathfrak{su}(2)^*$ and $*E$ is the two-form $E^a \epsilon_{abc} dx^b \wedge dx^c$.

The action of gauge transformations and diffeomorphisms on (A, E) induces a representation on cylindrical functions and fluxes. It is given by

$$(g \triangleright T)[A] = t(g(s_1)^{-1} h_{\alpha_1} g(e_1), g(s_2)^{-1} h_{\alpha_2} g(e_2), \dots) \quad (2.7)$$

$$(\phi \triangleright T)[A] = t(h_{\phi(\alpha_1)}[A], h_{\phi(\alpha_2)}[A], \dots) \quad (2.8)$$

$$(g \triangleright E_{S,f}) = E_{S, \text{ad}_g(f)} \quad (2.9)$$

$$(\phi \triangleright E_{S,f}) = E_{\phi(S), \phi_* f} \quad (2.10)$$

with t the function on $\text{SU}(2)^n$ characterizing the cylindrical functional T according to (2.5). To quantize cylindrical functions and fluxes, one is seeking a representation of the following algebraic relations on a Hilbert space:

$$\begin{aligned} T_1 \cdot T_2[A] &= T_1[A] T_2[A] \\ [T, E_{S,f}] &= 8\pi i l_p^2 X_{S,f}[T] \\ [T, [E_{S_1, f_1}, E_{S_2, f_2}]] &= (8\pi i l_p^2)^2 [X_{S_1, f_1}, X_{S_2, f_2}][T] \\ &\dots \\ (E_{S,f})^* &= E_{S, \bar{f}}, \quad (T[A])^* = \bar{T}[A] \end{aligned} \quad (2.11)$$

Here, X is a certain derivation on the space of cylindrical functions. These relations define an algebra. Diffeomorphisms and gauge transformations act as automorphisms.

The kinematical Hilbert space

$$\mathcal{H}_{\text{kin}} = L^2(\bar{\mathcal{A}}, \mu_{\text{AL}}). \quad (2.12)$$

is a space of functionals over the space of generalized connections $\bar{\mathcal{A}}$. The fluxes are represented by derivatives $X_{S,f}$,

$$\pi(E_{S,f}) = X_{S,f}. \quad (2.13)$$

Cylindrical functions act by multiplication. Gauge transformations and diffeomorphisms (and their semi-direct product) are unitarily implemented by the operators

$$\mathcal{U}_\phi |T\rangle = |\phi \triangleright T\rangle, \quad \mathcal{U}_g |T\rangle = |g \triangleright T\rangle. \quad (2.14)$$

This representation is called *Ashtekar-Lewandowski representation*. In the present article, we will also use the term *vacuum representation*, to contrast it with the new representations, which can be thought of as representations containing a “geometry condensate” [1].

Now we turn to the new representations [1]. Let $E^{(0)}$ be a classical triad field. Then, following Koslowski we define the new representation on *the same* Hilbert space as before, by changing the action of the fluxes:

$$\pi'(E_{S,f}) = X_{S,f} + E_{S,f}^{(0)} \text{id}, \quad \text{with } E_{S,f}^{(0)} = \int_S *E^{(0)} \lrcorner f_I. \quad (2.15)$$

It is easily checked that this gives another representation of the algebra. We list some elementary properties.

1. The spectra of the fluxes have changed. If λ is an eigenvalue of $X_{S,f}$ then $\lambda + E_{S,f}^{(0)}$ is an eigenvalue of $\pi'(E_{S,f})$. In particular, a constant cylindrical function is now an eigenfunction of $\pi'(E_{S,f})$ with in general non-vanishing eigenvalue $E_{S,f}^{(0)}$.
2. The new representation is still cyclic, with the empty spin net as cyclic vector.
3. The new representation is unitarily inequivalent to the standard one.
4. The operators implementing diffeomorphisms and gauge transformations in the standard representation are still well defined and unitary in the new representation, but it can be easily checked that they do not implement the algebra-automorphisms anymore. For example

$$U_\phi \pi'(E_{S,f}) U_{\phi^{-1}} \neq \pi'(\phi \triangleright E_{S,f}) \quad (2.16)$$

in general. The reason is that the U_ϕ do not change the background geometry.²

5. The standard kinematical representation can be viewed as a special case of the new one, for $E^{(0)} = 0$.

We note that point 5 is an impediment to the implementation of the gauge and diffeomorphism constraints. It is also the reason that the new representations are not counterexamples to the uniqueness theorems [4, 5], since the latter require the implementation of the action (2.7) – (2.10) on the basic operators.

We note again that in the new representations, the operators representing the connection remain untouched. One might wonder whether it is possible to obtain representations in which also a background connection is used. A straightforward way to proceed would be as follows:³ The new representations from above can be obtained by applying a translation

$$E \mapsto E + E^{(0)} \quad (2.17)$$

to the functionals $E_{S,f}$ of (2.6). A similar translation,

$$A \mapsto A + K^{(0)}, \quad (2.18)$$

is possible on the connection, where $K^{(0)}$ is a suitable fixed tensor field. The difference is that the translation on the triad field, when applied to the functional $E_{S,f}$, leads to a functional of the same type (flux plus constant term), whereas the translation on the connections turns a holonomy into a very complicated functional of the connection. Thus this translation leads out of the algebra of functionals that is the basis for loop quantum gravity and can thus not be used to define new states.

In the following section we will consider the existence and properties of the geometric operators for area and volume in the new representation. We then turn to the implementation of diffeomorphisms and gauge transformations, and to the implementation of the corresponding constraints.

²Indeed, one finds

$$U_\phi \pi'(E_{S,f}) U_{\phi^{-1}} = U_\phi \left(X_{S,f} + E_{S,f}^{(0)} \mathbb{I} \right) U_{\phi^{-1}} = X_{\phi(S), \phi_* f} + E_{S,f}^{(0)} \mathbb{I}$$

by virtue of the fact that the U_ϕ act on the $X_{S,f}$ in the standard way, but commute with the new c-number term.

³The following argument is due to J Lewandowski (private communication).

3 Geometric operators

In this section, we will show that area and volume operators can be defined in the new representations with exactly the same regularization and quantization strategy ([6, 7, 8]) as in the vacuum representation. This simplifies the treatment in [1], in which a new volume operator was used. Also, we extend the results about expectation values of the area operator obtained there. In fact, we make the structure of area and volume operator completely explicit. It is very simple. The action of the geometric operators in the new representations is given as a sum of two terms. One term is precisely the action in the vacuum representation, the other one is given as the value of the geometric quantity in the background geometry $E^{(0)}$ times the identity operator.

Proposition 3.1. *The standard regularization and quantization procedures for area and volume operators can be applied to the new representations and lead to well defined operators. We find*

$$V_R = V_R^{vac} + V_R^{(0)} \text{id}, \quad A_S = A_S^{vac} + A_S^{(0)} \text{id} \quad (3.1)$$

with V^{vac}, A^{vac} the geometric operators in the vacuum representation, and $V_R^{(0)}, A_S^{(0)}$ the classical values in the background geometry.

This result has a very simple form, but it is by no means trivial. A priori, coupling terms between geometry excitations and background could have appeared, or the regularization procedures could have broken down completely. We will have to work somewhat to exclude both possibilities. The result is remarkable in that it suggests a split into a background part (the c-number term) and fluctuations (represented by the vacuum operators), and thus a physical interpretation of the states in the new representations as excitations above a background spatial geometry. It should be stressed, however, that this split is a result of the application of the standard quantization procedure [6, 7, 8] to the new representations, not an assumption.

As we have already done in stating the result, from now on we will drop all the π 's indicating the representation. We hope that it is clear from context. Sometimes, we may add an index vac to denote an object in the vacuum representation, for clarity.

3.1 Area

Let us first consider the area operator. The whole calculation is modeled on [7], so we refer to that reference for further background. We pick a surface S and consider a family of non-negative densities $f_p^\epsilon(q)$ on S that has the property

$$\lim_{\epsilon \rightarrow 0} f_p^\epsilon(q) = \delta_p(q). \quad (3.2)$$

δ_p is the delta-function on S which is peaked at p . We also introduce the shorthands

$$E_\epsilon^I(q) = \pi(E_{S, b^I f_p^\epsilon(\cdot)}), \quad X_\epsilon^I(q) = X_{S, b^I f_p^\epsilon(\cdot)} \quad (3.3)$$

and

$$E^{(0)I}_\epsilon(p) = \int_S *E^{(0)I} f_p^\epsilon. \quad (3.4)$$

Then we can define

$$\Delta_\epsilon = E_\epsilon \cdot E_\epsilon. \quad (3.5)$$

As in [7], we would like to define

$$\hat{A}_S = \lim_{\epsilon \rightarrow 0} \int_S \sqrt{\Delta_\epsilon} \quad (3.6)$$

and study its properties. It can be seen immediately that Δ_ϵ is essentially self adjoint on cylindrical functions, and positive semidefinite. Therefore taking its square root is not a problem. Rather, what we have to worry about is the limit in (3.6). The structure of Δ_ϵ is

$$\Delta_\epsilon = X \cdot X + 2X \cdot E^{(0)} + E^{(0)} \cdot E^{(0)} \quad (3.7)$$

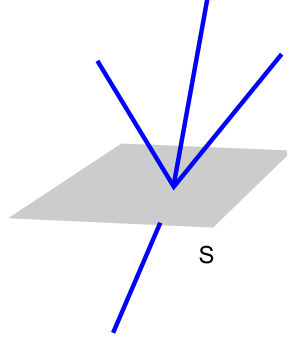


Figure 1: Surface with single vertex v as described in the text.

where the first term is the standard term. To see what the additional terms give, the first step is to go to a basis of eigenvectors for both, the $X \cdot X$ and the $X \cdot E^{(0)}$ term. We can consider the case of a single intersection vertex v (see figure 1), as the more general case can be handled by first subdividing the surface and considering the individual area operators separately. $X \cdot X$ and $X \cdot E^{(0)}$ can be diagonalized simultaneously. For a common eigenvector Ψ , one finds

$$X_\epsilon(p) \cdot X_\epsilon(p) \Psi = \alpha(f_p^\epsilon)^2(v) \Psi \quad (3.8)$$

and

$$X_\epsilon(p) \cdot E_\epsilon^{(0)}(p) \Psi = \beta |E_\epsilon^{(0)}(p)| f_p^\epsilon(v) \Psi. \quad (3.9)$$

Here \cdot and $|\dots|$ are with respect to the metric on $SU(2)$. The values α and β can obtain are determined by the representation labels on the edges intersecting at v . Thus we find for the regulated area operator

$$\int_S \sqrt{\Delta_\epsilon} \Psi = \int_S \left[\alpha(f_p^\epsilon)^2(v) + 2\beta |E^{(0)}_\epsilon(p)| f_p^\epsilon(v) + |E^{(0)}_\epsilon(p)|^2 \right]^{\frac{1}{2}} dp \Psi \quad (3.10)$$

$$= \int_S \left[\left(\sqrt{\alpha} f_p^\epsilon(v) + |E^{(0)}_\epsilon(p)| \right)^2 + (\beta - \sqrt{\alpha}) |E^{(0)}_\epsilon(p)| f_p^\epsilon(v) \right]^{\frac{1}{2}} dp \Psi \quad (3.11)$$

$$=: \int_S [a_\epsilon + b_\epsilon]^{\frac{1}{2}} dp \Psi. \quad (3.12)$$

To find the limit of the integral as $\epsilon \rightarrow 0$, we use the fact that for $a, b \geq 0$

$$\sqrt{a} \leq \sqrt{a+b} \leq \sqrt{a} + \sqrt{b}, \quad \sqrt{a} - \sqrt{b} \leq \sqrt{a-b} \leq \sqrt{a}, \quad (3.13)$$

where we have additionally assumed $b \leq a$ in the second pair of inequalities. We want to apply these inequalities to the integrand. a_ϵ is manifestly non-negative, but we do not know the sign of b_ϵ . In case it is negative, we still know that $a_\epsilon \geq |b_\epsilon|$ by virtue of the fact that the operator Δ_ϵ is non-negative. Thus the inequalities (3.13) apply, with $a = a_\epsilon, b = b_\epsilon$. To finish the calculation, we note that

$$\lim_{\epsilon \rightarrow 0} \sqrt{a_\epsilon} = \sqrt{\alpha} \lim_{\epsilon \rightarrow 0} \int_S f_p^\epsilon(v) dp + \lim_{\epsilon \rightarrow 0} \int_S |E^{(0)}_\epsilon(p)| dp = \sqrt{\alpha} + \int_S |E^{(0)}(p)| dp. \quad (3.14)$$

For $\sqrt{b_\epsilon}$ on the other hand, we note that the integrand is a product of $(f_p^\epsilon)^{1/2}$ and a bounded function. One can readily show that the integral therefore converges to zero,

$$\lim_{\epsilon \rightarrow 0} \sqrt{|b_\epsilon|} = 0. \quad (3.15)$$

Using the inequalities (3.13), we thus have

$$\lim_{\epsilon \rightarrow 0} \int_S \sqrt{\Delta_\epsilon} \Psi = \lim_{\epsilon \rightarrow 0} \sqrt{a_\epsilon} = \sqrt{\alpha} + \int_S |E^{(0)}(p)| dp. \quad (3.16)$$

The $\sqrt{\alpha}$ term is the one appearing in the vacuum representation, the second term is precisely the area of S in the background.

3.2 Volume

We will now turn to the volume operator. We refer to [8] for motivation and details about the chosen regularization. We will now describe the aspects of it that are relevant for our purpose. We are considering the volume of a region R . The regularization procedure consists in subdividing the region R into smaller and smaller cubes. For any given cube C , one defines three orthogonal surfaces $S_{i,C}$ and the phase space function

$$q_C[E] = \frac{1}{3!} \eta_{abc} \epsilon_{IJK} E_{S_a}^I E_{S_b}^J E_{S_c}^K \quad (3.17)$$

with η the totally anti-symmetric pseudo density. The volume is recovered as

$$V_R = \lim_C \sum \sqrt{|q_C|} \quad (3.18)$$

where the limit indicated is that of the subdivision into cubes getting finer and finer. This regularization leads the way to quantization, as q_C of (3.17) can immediately be quantized by replacing the classical flux quantities by their quantum counterparts. The limit in (3.18) is taken in such a way that the vertices of a graph underlying a state to be acted upon end up in the bulk and not on the boundary of the cubes. It turns out that once the subdivision is fine enough, the operator action on a given state stabilizes. Thus the limit down to cubes of coordinate-volume⁴ zero does not have to be carried out explicitly. In an additional step, the resulting operator is averaged in a certain way, to rid it of a remaining dependence on the subdivision and the positions of the surfaces inside the cubes.

In the vacuum representation, only cubes that contain a (three or higher valent) vertex contribute in the sum over cubes in (3.18). As we will show, this is no longer true in the new representations, and we will have to show that this does not lead to a blow-up of the sum in the limit of finer and finer subdivision. Indeed considering a fixed cube C , we will have

$$q_C[E] = \frac{1}{3!} \eta_{abc} \epsilon_{IJK} (X_{S_a}^I + E^{(0)I}_{S_a}) (X_{S_b}^J + E^{(0)J}_{S_b}) (X_{S_c}^K + E^{(0)K}_{S_c}), \quad (3.19)$$

thus apart from the X^3 term that was present in the vacuum representation, we also have terms $X^2 E^{(0)}$, $X(E^{(0)})^2$, and $(E^{(0)})^3$. To further analyze the situation, we consider states based on a graph γ and assume that the subdivision into cubes is already fine enough such that each cube in the sum in (3.18) is of one of the following types (see also figure 2):

0. A cube is of *type 0* if it is not intersected by any edge of the graph γ .
1. A cube is of *type 1* if it is intersected by one edge of the graph γ , and this intersection is in such a way that precisely one of the surfaces S_a internal to the cube is intersected transversally.
2. A cube is of *type 2* if it contains precisely one vertex, and such that precisely two of the surfaces S_a internal to the cube are intersected transversally by the edges emanating from the vertex.
3. A cube is of *type 3* if it contains precisely one vertex, and such that all three surfaces S_a internal to the cube are intersected transversally by the edges emanating from the vertex.

Thus we can write

$$V_R = \lim_C \sum_{i=0}^3 \sum_{C \text{ of type } i} \sqrt{|q_C|} \quad (3.20)$$

Let L^3 be the order of magnitude of the coordinate-volume of the region R , and ϵ^3 the typical coordinate-volume of a cube. When refining the subdivision into cubes, the total number N of

⁴Here and in the following, we denote with *coordinate volume* the volume with respect to some fiducial metric, for example the flat metric on \mathbb{R}^3 pulled back via a coordinate chart

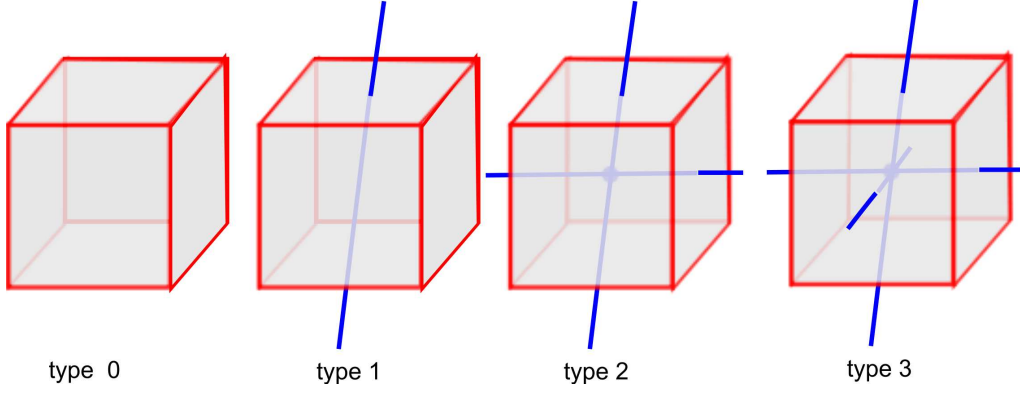


Figure 2: The different types of cubes considered in the text.

cubes in the sum in (3.18) thus goes like L^3/ϵ^3 . But the number of cubes of the different types behaves very differently: The number of cubes of type 0 make up the bulk and go as L^3/ϵ^3 . The number of cubes of type 1 goes as L/ϵ since the edges are one-dimensional objects. Finally there are only finitely many cubes of type 2 and 3, since there are only finitely many vertices.

Let us consider the action of q_C for C a cube of type 2 or 3: Schematically

$$q_C \Psi_\gamma = \sum \left(X^3 + X^2 E^{(0)} + X(E^{(0)})^2 + (E^{(0)})^3 \right) \Psi_\gamma \quad (3.21)$$

But $E^{(0)}$ goes as ϵ^2 , so we have

$$\lim q_C \Psi_\gamma = \lim \frac{1}{3!} \eta_{abc} \epsilon_{IJK} X_{S_a}^I X_{S_b}^J X_{S_c}^K \quad (3.22)$$

and thus, since the sum is finite

$$\lim \sum_{C \text{ of type 3,2}} \sqrt{|q_C|} \Psi_\gamma = V_R'^{(\text{old})} \Psi_\gamma \quad (3.23)$$

where $V_R'^{\text{vac}}$ is the volume operator from the vacuum representation, before the averaging to remove the remaining regularization dependence. Next we consider a cube C of type 0. We obviously have

$$q_C \Psi_\gamma = \frac{1}{3!} \eta_{abc} \epsilon_{IJK} E^{(0)I}_{S_a} E^{(0)J}_{S_b} E^{(0)K}_{S_c} \quad (3.24)$$

and hence, because of (3.18)

$$\lim \sum_{C \text{ of type 0}} \sqrt{|q_C|} \Psi_\gamma = V_R^{(0)} \Psi_\gamma \quad (3.25)$$

where $V_R^{(0)}$ is the volume in the background geometry. Finally, we will show that the sum over cubes of type 1 does not contribute in the limit. To that end, we note that for C of type 1, schematically

$$q_C \Psi_\gamma = \sum_{C \text{ of type 1}} \left(X(E^{(0)})^2 + (E^{(0)})^3 \right) \Psi_\gamma. \quad (3.26)$$

Since $E^{(0)}$ is going like ϵ^2 , we have that $\epsilon^{-2} \sqrt{|q_C|}$ is converging to a finite operator. But then

$$\sum_{C \text{ of type 1}} \sqrt{|q_C|} \Psi_\gamma = \epsilon^2 \sum_{C \text{ of type 1}} \frac{1}{\epsilon^2} \sqrt{|q_C|} \Psi_\gamma \quad (3.27)$$

is a sum over $\approx L/\epsilon$ vectors, each with norm of order ϵ^2 , and hence this sum converges to zero in the limit of ϵ going to zero. Thus we have proven that

$$\lim \sum_{C \text{ of type 3,2}} \sqrt{|q_C|} + \lim \sum_{C \text{ of type 0}} \sqrt{|q_C|} = V_R'^{\text{vac}} + V_R^{(0)}. \quad (3.28)$$

as desired. Finally, we have to carry out the averaging procedure, but this is now trivial. The end result is a volume operator

$$\lim_{C \text{ of type } 3,2} \sum \sqrt{|q_c|} + \lim_{C \text{ of type } 0} \sum \sqrt{|q_c|} = V_R^{\text{vac}} + V_R^{(0)}. \quad (3.29)$$

with V_R^{vac} being the well known volume operator in the vacuum representation.

4 Unitary implementation of bundle automorphisms

Now we consider the implementation of the automorphisms of the $SU(2)$ principal fiber bundle underlying the whole formalism. We will follow exactly the treatment by Koslowski [1]. As is customarily done, we discuss the automorphisms induced by diffeomorphisms and the gauge transformations separately. Also in this section (and in all the following as well), we will drop all the π 's indicating the representation. We hope that it is clear from context in which representation we work.

The first observation is that the unitary operators that implement diffeomorphisms and gauge transformations in the vacuum representation are still well defined and unitary in the new representations. The problem is that they do not implement those transformations anymore. For a gauge transformation g and a diffeomorphism ϕ we have

$$U_\phi E_{S,f} U_\phi^{-1} = X_{\phi(S), \phi^{-1}*f} + E_{S,f}^{(0)} \neq \phi \triangleright E_{S,f} \quad (4.1)$$

$$U_g E_{S,f} U_g^{-1} = X_{S, \text{ad}_g(f)} + E_{S,f}^{(0)} \neq g \triangleright E_{S,f}. \quad (4.2)$$

Vice versa, it can be shown that if families of operators $\{U_g\}$ and $\{U_\phi\}$ implement the gauge transformations and diffeomorphisms, respectively, most of those operators can not be unitary.

The problem is evidently that the background geometry present in the new representations is immutable by operations on the Hilbert space and thus prevents unitary implementations of diffeomorphisms and gauge transformations. This problem can be solved by enlarging the Hilbert space in such a way that also the background geometries can be transformed, as was already observed in [1].

For a given background $E^{(0)}$ we consider the Hilbert space

$$\mathcal{H}_{[E^{(0)}]} := \bigoplus_{\tilde{E}^{(0)} \in [E^{(0)}]} \mathcal{H}_{\tilde{E}^{(0)}} \quad (4.3)$$

where $[E^{(0)}]$ is the equivalence class of $E^{(0)}$ with respect to diffeomorphisms and gauge transformations. The Hilbert space is thus effectively labeled by a spatial background metric modulo diffeomorphisms. We will write $|T, \tilde{E}^{(0)}\rangle$ for the cylindrical state T in the (new) representation with background geometry $\tilde{E}^{(0)}$ in the above Hilbert space. Thus we have

$$\langle T, E^{(0)} | T', E'^{(0)} \rangle = \langle T | T' \rangle \delta_{E^{(0)}, E'^{(0)}} \quad (4.4)$$

where the first scalar product on the right hand side is just the one in the vacuum representation. The Hilbert space $\mathcal{H}_{[E^{(0)}]}$ carries a representation of the LQG algebra, which is simply the direct sum of the representations on the individual spaces. But now one can implement the diffeomorphisms and gauge transformations unitarily, by setting [1]

$$U_\phi |T, E^{(0)}\rangle = |\phi \triangleright T, \phi \triangleright E^{(0)}\rangle, \quad U_g |T, E^{(0)}\rangle = |g \triangleright T, g \triangleright E^{(0)}\rangle. \quad (4.5)$$

We have to check that this definition gives a unitary representation, and implements the transformations on operators correctly. That it is a representation follows from the fact that the various actions \triangleright involved are all representations. That it is unitary is immediately seen from

the fact that both, the vacuum scalar product, and the Kronecker delta used in (4.4) are invariant under gauge transformations and diffeomorphisms.

It is easy to show that with this definition, the transformations are also correctly implemented on the operators. We give the example of the diffeomorphisms. We calculate

$$U_\phi E_{S,f} |T, E^{(0)}\rangle = U_\phi \left[|X_{S,f} T, E^{(0)}\rangle + E_{S,f}^{(0)} |T, E^{(0)}\rangle \right] \quad (4.6)$$

$$= |(\phi \triangleright X_{S,f})(\phi \triangleright T), \phi \triangleright E^{(0)}\rangle + E_{S,f}^{(0)} |\phi \triangleright T, \phi \triangleright E^{(0)}\rangle \quad (4.7)$$

$$= \phi \triangleright E_{S,f} |\phi \triangleright T, \phi \triangleright E^{(0)}\rangle \quad (4.8)$$

where in the last step we have used the fact that

$$\phi \triangleright E_{S,f} |\phi \triangleright T, \phi \triangleright E^{(0)}\rangle = |(\phi \triangleright X_{S,f})(\phi \triangleright T), \phi \triangleright E^{(0)}\rangle + \int_{\phi(S)} (\phi_* f)^I (\phi_* E^{(0)})_I |\phi \triangleright T, \phi \triangleright E^{(0)}\rangle \quad (4.9)$$

and that the integral simply evaluates to $E_{S,f}^{(0)}$ since both, integrand and integration region are pushed forward with ϕ . From this we read off, that

$$U_\phi E_{S,f} U_\phi^{-1} = \phi \triangleright E_{S,f}. \quad (4.10)$$

For a cylindrical function, the calculation is even simpler:

$$U_\phi T' |T, E^{(0)}\rangle = U_\phi |T' T, E^{(0)}\rangle = |\phi \triangleright (T' T), \phi \triangleright E^{(0)}\rangle \quad (4.11)$$

$$= |(\phi \triangleright T')(\phi \triangleright T), \phi \triangleright E^{(0)}\rangle \quad (4.12)$$

$$= (\phi \triangleright T') |\phi \triangleright T, \phi \triangleright E^{(0)}\rangle \quad (4.13)$$

from which we read off

$$U_\phi T' U_\phi^{-1} = \phi \triangleright T'. \quad (4.14)$$

A similar calculation can be carried out for the gauge transformations, showing that they, too, are implemented on operators correctly.

Finally, a straightforward calculation shows that not only the groups of gauge transformations and spatial diffeomorphisms are correctly implemented, but also their direct product (2.1). If one sets $U_{(g,\phi)} := U_g U_\phi$ one finds

$$U_{(g,\phi)} U_{(g',\phi')} = U_{(g,\phi) \cdot (g',\phi')}. \quad (4.15)$$

We have thus shown that the action of the bundle automorphisms on the basic variables of the theory are represented in an anomaly-free way. This in turn means that the hypersurface-algebra corresponding to the bundle automorphisms is represented in an anomaly-free way.⁵ The situation with respect to the constraint algebra is thus precisely the same as in the vacuum representation, and we are set to tackle the implementations of the constraints with the same methods as in the vacuum case.

In (4.3), the subspaces corresponding to different backgrounds are specified to be orthogonal to each other. The reader may wonder whether a different choice would be permissible, such that the inner product between states with the same background is unchanged, but states with different backgrounds are not necessarily orthogonal to each other. It is however easy to see that the above choice is the only one that renders $E_{S,f}$ self-adjoint.

We finally remark that the representation of the kinematical algebra of loop quantum gravity on the Hilbert space $\mathcal{H}_{[E^{(0)}]}$ is not a counter example to the uniqueness results [4, 5], despite the fact that it carries a unitary implementation of the bundle automorphisms. The theorems require a cyclic representation, but the representation on $\mathcal{H}_{[E^{(0)}]}$ is highly reducible, the direct sum decomposition in (4.3) giving a decomposition into invariant subspaces.

⁵One should not take this statement too literally though, since – as in the vacuum representation – the operators generating the unitary transformations are not well defined.

5 Implementation of the constraints

Now we are looking at the implementation of the constraints. The natural expectation would be that implementing the constraints using the new representations is not possible, since they contain background geometry. But this is not the case, as was realized in [1]. In fact, the diffeomorphism constraint can be implemented exactly as in the vacuum-representation. The situation is slightly more complicated for the Gauß constraint, as we will explain below. With hindsight, it is not too surprising that the implementation of the constraints succeeds. Implementing them is not about ridding the states of background *per se*, but ridding them of coordinate and gauge dependence.

We will first consider the implementation of the diffeomorphism constraint. Then we will turn to the Gauß constraint. Finally we will consider states that satisfy both classes of constraints.

5.1 Diffeomorphism constraint

The diffeomorphism constraint can be implemented on vectors in $\mathcal{H}_{[E^{(0)}]}$ with *exactly* the same strategy as in the vacuum representation. As far as we understand, this is also the strategy used in [1], but we will spell out more details. We will borrow notation from [9] and urge the reader to consult the reference for additional information about the standard treatment.

To start with, we make some definitions. We will call a diffeomorphism ϕ a symmetry of a triad $E^{(0)}$, if ϕ leaves E invariant, $\phi \triangleright E^{(0)} = E^{(0)}$ ⁶. Then

1. Let Diff be the group of all diffeomorphisms.
2. Let $\text{Diff}_{(\alpha, E^{(0)})}$ be the group of diffeomorphisms that are symmetries of $E^{(0)}$ and map the graph α onto itself. Note: If $E^{(0)} = 0$ these are just the diffeomorphisms mapping α onto itself (denoted Diff_α) in [7].
3. Let $\text{TDiff}_{(\alpha, E^{(0)})}$ be the group of diffeomorphisms that are symmetries of $E^{(0)}$ and map each edge of the graph α onto itself. Note: Again, if $E^{(0)} = 0$ these are just the diffeomorphisms mapping each edge of α onto itself, denoted as TDiff_α in [7].
4. Let $\text{Symm}_{(\alpha, E^{(0)})}$ be the quotient $\text{Diff}_{(\alpha, E^{(0)})} / \text{TDiff}_{(\alpha, E^{(0)})}$.

Let T_α be a cylindrical function that depends on the holonomies of all the edges of α in a non-trivial way. We can then define the linear form $(T_\alpha, E^{(0)})$ on $\mathcal{H}_{[E^{(0)}]}$ as follows:

$$(T_\alpha, E^{(0)} | T_\beta, E'^{(0)}) := \sum_{[\phi] \in \text{Diff} / \text{Diff}_{(\alpha, E^{(0)})}} \langle T_\alpha, E^{(0)} | P_{(\alpha, E^{(0)})}^\dagger U_\phi^\dagger | T_\beta, E'^{(0)} \rangle \quad (5.1)$$

where the projection $P_{(\alpha, E^{(0)})}$ is defined as

$$P_{(\alpha, E^{(0)})} | T_\alpha, E^{(0)} \rangle := \frac{1}{|\text{Symm}_{(\alpha, E^{(0)})}|} \sum_{[\phi] \in \text{Symm}_{(\alpha, E^{(0)})}} U_\phi | T_\alpha, E^{(0)} \rangle. \quad (5.2)$$

Now we have to demonstrate that these definitions makes sense, i.e. that they are well defined and give finite results, and that $(T_\alpha, E^{(0)})$ is indeed a diffeomorphism invariant state.

We will first address the issue of the the notions being well defined: If we chose, instead of ϕ a different member ϕ' of the equivalence class $[\phi]$ on the right hand side of (5.2), we will have $\phi' = \phi\omega$, where ω is in $\text{TDiff}_{(\alpha, E^{(0)})}$. But then $U_{\phi'} | T_\alpha, E^{(0)} \rangle = U_\phi | T_\alpha, E^{(0)} \rangle$ because ω leaves the state invariant by definition. When we chose a different representative ϕ' on the right hand side of (5.1), this can be equivalently written as using the original ϕ , but then $\theta\phi$ in place of ϕ

⁶This is closely related to the concept of isometry: An isometry of the spatial geometry would be a diffeomorphism that leaves $E^{(0)}$ invariant up to a gauge transformation. These will be relevant in section 5.3

in (5.2), where θ is in $\text{TDiff}_{(\alpha, E^{(0)})}$. But it is immediately shown, that this does not change the result of $P_{(\alpha, E^{(0)})}$. Finally, it is easy to show that, due to the division by the size of $\text{Symm}_{(\alpha, E^{(0)})}$, the above is invariant under subdivision of edges of α .

The next comment is on finiteness. We first show:

Lemma 5.1. $\text{Symm}_{(\alpha, E^{(0)})}$ is finite.

Proof. $\text{Diff}_{(\alpha, E^{(0)})}$ is a subset of Diff_α . Therefore elements of $\text{Symm}_{(\alpha, E^{(0)})}$ will consist of elements of Diff_α . We know that $\text{Diff}_\alpha / \text{TDiff}_\alpha$ is finite. We ask: Can two elements ϕ, ϕ' of $\text{Diff}_{(\alpha, E^{(0)})}$ that are in the same class in $\text{Diff}_\alpha / \text{TDiff}_\alpha$ be in different classes in $\text{Symm}_{(\alpha, E^{(0)})}$? If the answer is 'no', then $\text{Symm}_{(\alpha, E^{(0)})}$ can have at most as many classes as $\text{Diff}_\alpha / \text{TDiff}_\alpha$, and the lemma is proven. To show that this is indeed the case, take ϕ, ϕ' in $\text{Diff}_{(\alpha, E^{(0)})}$ and in the same class in $\text{Diff}_\alpha / \text{TDiff}_\alpha$, i.e.

$$\phi = \phi' \tilde{\phi}, \quad \tilde{\phi} \in \text{TDiff}_\alpha. \quad (5.3)$$

but then $\tilde{\phi} = (\phi')^{-1} \phi$, and $\text{Diff}_{(\alpha, E^{(0)})}$ is a group, so $\tilde{\phi}$ is also in $\text{Diff}_{(\alpha, E^{(0)})}$. Thus classes in $\text{Diff}_\alpha / \text{TDiff}_\alpha$ are not split over classes in $\text{Symm}_{(\alpha, E^{(0)})}$, and thus the lemma is proven. \square

Thus $P_{(\alpha, E^{(0)})}$ is a finite projection. Finally we remark that while the sum in (5.1) is infinite, only finitely many terms are non-zero, due to the fact that the sum is effectively only over diffeomorphisms that act non-trivial on the state $|T_\alpha, E^{(0)}\rangle$, and the scalar product is only nonzero, if the backgrounds and graphs in the scalar product coincide *exactly*.

Finally, let us demonstrate that the functional $(T_\alpha, E^{(0)})$ is indeed diffeomorphism invariant. To this end, consider

$$(T_\alpha, E^{(0)} | U_\omega | T_\beta, E'^{(0)}) \quad (5.4)$$

where ω is an arbitrary diffeomorphism. If we write out the sums, we get an expression of the form

$$\sum_{[\phi] \in \text{Diff} / \text{Diff}_{(\alpha, E^{(0)})}} \sum_{[\phi'] \in \text{Symm}_{(\alpha, E^{(0)})}} \langle \dots | U_\phi^\dagger U_{\phi'}^\dagger U_\omega | \dots \rangle. \quad (5.5)$$

But since Diff is a group and $\text{Diff}_{(\alpha, E^{(0)})}$ acts from the right, we have that

$$\sum_{[\phi] \in \text{Diff} / \text{Diff}_{(\alpha, E^{(0)})}} U_{\omega^{-1} \circ \phi}^\dagger = \sum_{[\phi] \in \text{Diff} / \text{Diff}_{(\alpha, E^{(0)})}} U_\phi^\dagger \quad (5.6)$$

and hence

$$(T_\alpha, E^{(0)} | U_\omega | T_\beta, E'^{(0)}) = (T_\alpha, E^{(0)} | T_\beta, E'^{(0)}) \quad (5.7)$$

as it should be.

The definition of the inner product on the diffeomorphism invariant states now proceeds exactly as in the vacuum case, and leads to a diffeomorphism invariant Hilbert space $\mathcal{H}_{[E^{(0)}]}^{\text{Diff}}$. We emphasize that the construction is such that for the fully degenerate background geometry $E^{(0)} = 0$, we have $\mathcal{H}_{[0]}^{\text{Diff}} = \mathcal{H}^{\text{Diff}}$, where the latter is the (standard) diffeomorphism-invariant Hilbert space of the vacuum representation.

5.2 Gauß constraint

Now we come to the implementation of the Gauß constraint. Here our treatment will differ somewhat from [1]. We will comment on the differences below.

Also, we find that we cannot blindly follow the strategy used in the vacuum representation: There, implementation of the constraint can be done by group averaging involving an *integral* over the group of gauge transformations. While this group is huge, since there are only finitely many vertices involved in any given graph, the integral boils down to finitely many integrals

over $SU(2)$ and is thus well defined. In our context, since gauge transformation may act non-trivially on all of $E^{(0)}$, it is unclear how to extend this treatment to the new representations. Instead one might be tempted to follow the strategy that was used in the case of the diffeomorphism constraint. There, a *sum* over the group elements was performed. Now, since a gauge transform of a state may not be orthogonal to the state itself in the vacuum representation, one might suspect that this strategy runs into problems with convergence. Indeed, this is the case. To see this, we try to define

$$\{T, E^{(0)}| = \sum_{g \in \mathcal{G} / \mathcal{G}_{T, E^{(0)}}} \langle T, E^{(0)} | U_g^\dagger \quad (5.8)$$

where $\mathcal{G}_{T, E^{(0)}}$ are the gauge transformation that are a symmetry of $E^{(0)}$ (i.e. leave $E^{(0)}$ invariant) and leave T invariant. The problem with this is that the sum may contain infinitely many g which change T but leave $E^{(0)}$ invariant. Think for example of gauge transformations that are rotations with axis $E^{(0)}$ at each point. Such transformations will still generically change a given functional T . When evaluating $\{T, E^{(0)}|$ on a state $|T', E^{(0)}\rangle$ with T' gauge invariant and $\langle T | T' \rangle \neq 0$, each transformation that fixes $E^{(0)}$ but changes T will produce the same non-vanishing number, leading to a divergence.

The problem described above can be avoided if one starts with gauge invariant functionals T . Then obviously there are no gauge transformations that change T and leave $E^{(0)}$ invariant. Let $\mathcal{G}_E^{(0)}$ be the group of gauge transformations that are symmetries of $E^{(0)}$. For a gauge invariant T we set

$$\{T, E^{(0)}| := \sum_{g \in \mathcal{G} / \mathcal{G}_E^{(0)}} \langle T, E^{(0)} | U_g^\dagger. \quad (5.9)$$

This is obviously gauge invariant, and evaluated on a state there can be at most one non-zero term:

$$\{T, E^{(0)} | T', E'^{(0)}\rangle = \langle T | T' \rangle \begin{cases} 1 & \text{if } E^{(0)}, E'^{(0)} \text{ are gauge-related} \\ 0 & \text{otherwise} \end{cases} \quad (5.10)$$

We emphasize that the construction is such that for the fully degenerate background geometry $E^{(0)} = 0$, we have $\mathcal{H}_{[0]}^{\mathcal{G}} = \mathcal{H}^{\mathcal{G}}$, where the latter is the (standard) gauge-invariant Hilbert space of the vacuum representation.

We also note that (in view of (5.10)), an orthonormal basis of $\mathcal{H}_{[E^{(0)}]}^{\mathcal{G}}$ can be given as follows: First, since $|T, E^{(0)}\rangle$ depends on $E^{(0)}$ only through the gauge equivalence class $[E^{(0)}]$, we might as well denote it by $|T, [E^{(0)}]\rangle$. Then a basis is given by $\{|T, [E^{(0)}]\rangle\}$, where T runs through all spin nets, and $[E^{(0)}]$ through all gauge equivalence classes.

We now come to the difference with [1]. Koslowski puts forward the idea, that one should also consider states in which the connection is coupled to the background, as these can be gauge invariant, too. Take for example the case of a loop α with endpoint p , and consider the functional

$$T[A, E^{(0)}] = \text{tr} \left(h_\alpha[A] E^{(0)}(p) \right). \quad (5.11)$$

This is invariant, provided both A and $E^{(0)}$ transform under gauge transformations. But we do not see how this can be achieved in the present formalism: Nothing prevents us from defining the state $|T[\cdot, E^{(0)}], E^{(0)}\rangle$, where T is the functional defined above, but it is not gauge invariant. Rather

$$U_g |T[\cdot, E^{(0)}], E^{(0)}\rangle = |T[g^{-1} \cdot g, E^{(0)}], g \triangleright E^{(0)}\rangle = |T[\cdot, g \triangleright E^{(0)}], g \triangleright E^{(0)}\rangle. \quad (5.12)$$

Instead, one might consider it more promising to start with the functional (a treatment along these lines was probably implied in [1])

$$\{\psi| := \sum_{g \in \mathcal{G} / \mathcal{G}_{0, E^{(0)}}} \langle T[\cdot, E^{(0)}], g \triangleright E^{(0)}|. \quad (5.13)$$

Here, 0 denotes the zero spin network. One can envision this functional as being obtained from the action of a hypothetical operator $\text{tr}(h_\alpha \hat{E}^{(0)}(p))$ on the state

$$\{0, [E^{(0)}]\} \equiv \sum_{g \in \mathcal{G}/\mathcal{G}_{E^{(0)}}} \langle 0, g \triangleright E^{(0)} |. \quad (5.14)$$

Indeed, operators of this type have been considered in the early days of loop quantum gravity. One can check that (5.13) defines functional on states,

$$\{\psi | T, E'^{(0)}\} = \begin{cases} \langle T[\cdot, E'^{(0)}] | T \rangle & \text{if } E'^{(0)} \in [E] \\ 0 & \text{otherwise} \end{cases}. \quad (5.15)$$

A straightforward calculation shows that this functional is also gauge invariant. It is, however, not an element of the diffeomorphism invariant Hilbert space. This can be seen very easily as follows: While $\{\psi |$ evaluates to something non-zero only on states $|T, E'^{(0)}\rangle$ where T is not gauge invariant, all functionals in $\mathcal{H}_{[E^{(0)}]}^{\mathcal{G}}$ are non-zero only on states $|T, E'^{(0)}\rangle$ where T is gauge invariant.

Thus, to give functionals such as (5.15) a home, one would need a scalar product on them. This may be possible, for example by modifying the procedure suggested in (5.8) in such a way that the sum is over a smaller set. This however risks to yield a non-gauge-invariant functional. To summarize, it may well possible – and this possibility is very exciting – to also introduce states that couple to the background, but this would necessitate a departure from the framework considered here (and as far as we understand, also from the one in [1]).

5.3 States invariant under bundle automorphisms

Up to now we have considered diffeomorphisms and gauge transformations separately. Now we want to show how one can obtain states that are invariant under their semidirect product, the bundle automorphisms. The idea is to first average over gauge transformations, then over diffeomorphisms. One has to be careful, however because there may be diffeomorphisms that are symmetries up to a gauge transformation. Applying these to a gauge invariant functional and summing over them leads to an over-counting and, possibly, to a divergence.

Thus the averaging for the diffeomorphism invariant states has to be modified slightly. We denote by $\text{Giff}_{(\alpha, E^{(0)})}$ the group of diffeomorphisms that are symmetries of $E^{(0)}$ up to a gauge transformation, and map the graph α onto itself. Let $\text{TGiff}_{(\alpha, E^{(0)})}$ be the group of diffeomorphisms that are symmetries of $E^{(0)}$ up to gauge transformations and map each edge of the graph α onto itself. Finally Let $\text{Gymm}_{(\alpha, E^{(0)})}$ be the quotient of the above. Then, for a state $|T, E^{(0)}\rangle$ with T gauge invariant, the group averaging over diffeomorphisms and gauge transformations is

$$(T_\alpha, E^{(0)} | T_\beta, E'^{(0)}) := \sum_{[\phi] \in \text{Diff} / \text{Giff}_{(\alpha, E^{(0)})}} \{T_\alpha, E^{(0)} | P_{(\alpha, E^{(0)})}^\dagger U_\phi^\dagger | T_\beta, E'^{(0)}\} \quad (5.16)$$

where

$$P_{(\alpha, E^{(0)})} | T_\alpha, E^{(0)} \rangle := \frac{1}{|\text{Gymm}_{(\alpha, E^{(0)})}|} \sum_{[\phi] \in \text{Gymm}_{(\alpha, E^{(0)})}} U_\phi | T_\alpha, E^{(0)} \rangle. \quad (5.17)$$

It is easy to show that this is well defined, with the same reasoning as given in section 5.1. The proof that Gymm is finite goes through exactly as before. It is also easy to see that evaluation is always finite. Finally it is obvious that the resulting functional is invariant under both, diffeomorphisms and gauge transformations.

6 Outlook

In the present article, we have demonstrated how the new representations form a family with the (standard) vacuum representation being one member. This begs the question as to the significance and use of these representations.

As we have already stated before, we do not think that the new representations are as fundamental as the vacuum representation, because they contain so much classical background structure. Still, it may be interesting to see in what sense they can be used to solve the Hamiltonian constraint. On the one hand, one could take over Thiemann's construction literally, and see whether the constraint is well defined, and how solutions would look. On the other hand the new representations allow for new quantization strategies, because the volume operator can now have non-vanishing kernel. Likewise, we have preliminary indications that one may be able to define an operator corresponding to the integral of curvature over a surface.

All of this would also be interesting for applying the new representation in situations, in which one would like to treat most of the gravitational excitations as background, and only study in detail the excitations of geometry over this kind of geometric condensate.

Finally, the new representations could be a spring board to the construction of yet other representations which are more semiclassical.

On a more technical level, it may be interesting to study different types of backgrounds. When we were discussing the backgrounds $E^{(0)}$ the reader may have had a smooth, non-degenerate geometry in mind. But more general choices are also possible, as long as fluxes, areas and volumes are still well defined. Possibilities include for example fields that are only non-vanishing on submanifolds.

We hope to come back to some of the questions raised above, at another time.

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